3) Solutions for Motion in 1D

3.1) Motion with constant Force

Let the force be given by

$$\vec{F} = F\hat{i}$$

where $\underline{F}$ is a $\underline{\text{constant}}$.

The equation of motion is then

$$m\ddot{x}\hat{i} = F\hat{i} \Rightarrow \boxed{\begin{aligned} &m\ddot{x} = F \\ \Rightarrow\ &\ddot{x} = \frac{F}{m} \end{aligned}}$$

where F/m is also a constant.

$\ddot{x} = \dfrac{F}{m}$ is a linear ODE.

We supplement the ODE with 2 initial conditions (for position and velocity):

$$\dot{x}(0) = v_0 \ , \quad x(0) = x_0$$

where the initial position and initial velocity is given.

The ODE can be solved in 2 equivalent ways by integrating it twice in variable t:

Method 1 Using indefinite integrals:

Integrating the ODE, with respect to t

$$\frac{d^2x}{dt^2} = \frac{F}{m} \Rightarrow \int \frac{d^2x}{dt^2}\,dt = \frac{F}{m}\int dt$$

$$\Rightarrow \frac{dx}{dt} = \frac{Ft}{m} + c_1$$

and

$$\frac{dx}{dt} = \frac{Ft}{m} + c_1 \Rightarrow \int \frac{dx}{dt}\,dt = \int \frac{Ft}{m} + c_1\,dt$$

$$\Rightarrow x = \frac{F}{m}\frac{t^2}{2} + c_1 t + c_2$$

where c_1 and c_2 are arbitrary constants of integration.

Use boundary conditions / initial conditions to determine c_1 & c_2

$$\dot{x}(0) = v_0 \Rightarrow \frac{F}{m}(0) + c_1 = v_0$$

$$\Rightarrow c_1 = v_0$$

$$x(0) = x_0 \Rightarrow \frac{F}{m}(0)^2 + c_1(0) + c_2 = x_0$$

$$\Rightarrow c_2 = x_0$$

Therefore solution to equation of motion is

$$x(t) = x_0 + v_0 t + \frac{F}{m}\frac{t^2}{2}$$

Method 2: (definite integral)

Here we integrate the equation of motion from 0 to t

$$\int_0^t \ddot{x}(s)\,ds = \int_0^t \frac{F}{m}\,ds$$

$\Rightarrow$

$$\dot{x}(t) - \dot{x}(0) = \frac{F}{m}t - \frac{F}{m}(0)$$

$$\Rightarrow \dot{x}(t) = \dot{x}(0) + \frac{F}{m}(t)$$

(note the variable of integration has been renamed to distinguish it from the upper limit of integration t)

$$\Rightarrow \int_0^t \dot{x}(s)\,ds = \int_0^t \dot{x}(0) + \frac{F}{m}.s \; ds$$

$$\Rightarrow x(t) - x(0) = \left(\dot{x}(0)t + \frac{F}{m}\frac{t^2}{2}\right) - \left(\dot{x}(0).0 + \frac{F}{m}\frac{0^2}{2}\right)$$

$$\Rightarrow x(t) = x(0) + \dot{x}(0)t + \frac{F}{m}\frac{t^2}{2}$$

$$\Rightarrow x(t) = x_0 + v_0 t + \frac{F}{m}\frac{t^2}{2}$$

as the general solution.

Example problem 1: (uniform gravity force)

Consider the motion of a body of mass m under the action of the gravity force.

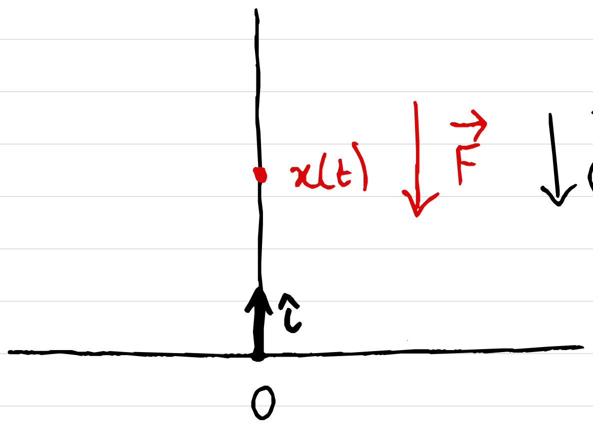
Initial height: $x(0) = x_0$ above earth surface

Initial velocity: $v(0) = v_0$ moving upwards

↳ vertical velocity

(i) Find maximum height body will reach

Solution: Let height above earth surface at time t be $x(t)$



Co-ordinate Ox is vertical and directed upwards.

The only force on the body is gravity being applied downwards. Since the net force and gravity are in the same direction; The equation of motion is given by

$$m\vec{\ddot{x}} = m\vec{g} \Rightarrow \vec{\ddot{x}} = \vec{g}$$

Since $\vec{g} = -g\hat{i}$

$$\vec{\ddot{x}} = \vec{g} \Rightarrow \ddot{x}\hat{i} = -g\hat{i}$$

which gives scalar eqn

$$\ddot{x} = -g$$

Solving ODE $\ddot{x} = -g$

$$\ddot{x} = -g \Rightarrow \int_0^t \ddot{x}(s)\,ds = \int_0^t -g\,ds$$

$$\Rightarrow \dot{x}(t) - \dot{x}(0) = -gt - g(0)$$

$$\Rightarrow \dot{x}(t) = \dot{x}(0) - gt$$

Solving $\dot{x}(t) = \dot{x}(0) - gt$

$$\dot{x}(t) = \dot{x}(0) - gt \Rightarrow \int_0^t \dot{x}(s)\,ds = \int_0^t \dot{x}(0) - gs\,ds$$

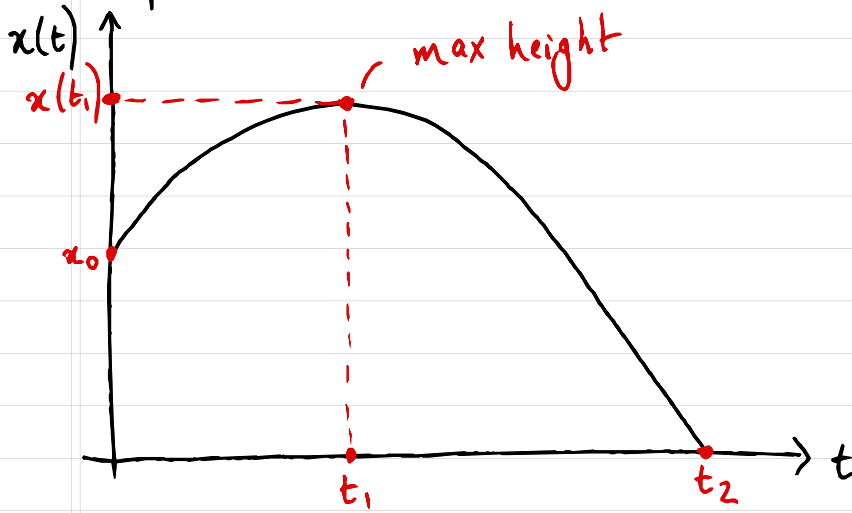
$$\Rightarrow x(t) - x(0) = \dot{x}(0)t - \frac{gt^2}{2}$$

$$\Rightarrow x(t) = x(0) + \dot{x}(0)t - \frac{gt^2}{2}$$

So the general solution is

$$x(t) = x_0 + v_0 t - \frac{gt^2}{2}$$

Position x as a function of t is part of a parabola.



Let t_1 be the time when body reaches max height.

This point is a maximum of the graph $x(t)$ i.e

$$\dot{x}(t) = \frac{dx}{dt} = v(t) = 0$$

Therefore

$$v_0 - gt_1 = 0 \Rightarrow t_1 = \frac{v_0}{g}$$

$$x(t_1) = x_0 + v_0 t_1 - g\frac{t_1^2}{2}$$

$$= x_0 + v_0 \cdot \frac{v_0}{g} - \frac{\cancel{g}}{2} \cdot \frac{v_0^2}{g^{\cancel{2}}}$$

$$= x_0 + \frac{v_0^2}{2g}$$

(ii) Time when it will fall to the ground

Let t_2 be the time when ball touches the ground.

At that moment:

$$x(t_2) = 0$$

$$x(t_2) = 0 \Rightarrow x_0 + v_0 t_2 - \frac{g t_2^2}{2} = 0$$

Solving quadratic formula for t_2,

$$t_2 = \frac{v_0}{g} + \sqrt{\frac{v_0^2}{g} + \frac{2x_0}{g}}$$

(exclude negative root, irrelevant to problem)

3.2) Motion with a time-dependant force

If

$$\vec{F} = F(t)\hat{\imath}$$

where $\underline{F(t)}$ is a $\underline{\text{given function of time}}$ then we have

$$\vec{F} = F(t)\hat{\imath} \Rightarrow \boxed{\begin{aligned} m\ddot{x}\hat{\imath} &= F(t)\hat{\imath} \\ \Rightarrow m\ddot{x} &= F(t) \end{aligned}}$$

Lets solve ODE with initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

Integrating the equation of motion from 0 to t,

$$\int_0^t \ddot{x}(s)\,ds = \frac{1}{m}\int_0^t F(s)\,ds$$

$$\Rightarrow$$

$$\dot{x}(t) - \dot{x}(0) = \frac{1}{m}\int_0^t F(s)\,ds$$

$$\Rightarrow$$

$$\dot{x}(t) = x(0) + \frac{1}{m}\int_0^t F(s)\,ds$$

Yet another integration yeilds

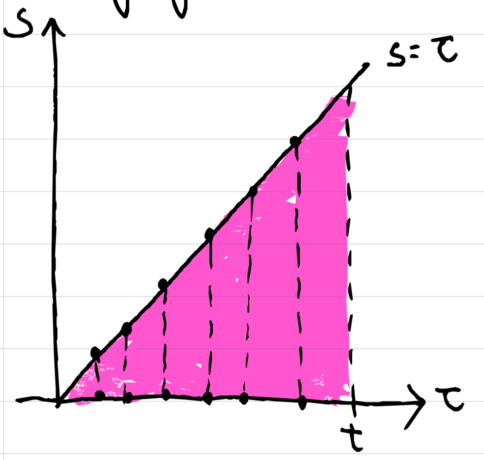
$$\int_0^t \dot{x}(\tau)\,d\tau = \int_0^t \dot{x}(0)\,d\tau + \frac{1}{m}\int_0^t \left(\int_0^\tau F(s)\,ds\right) d\tau$$

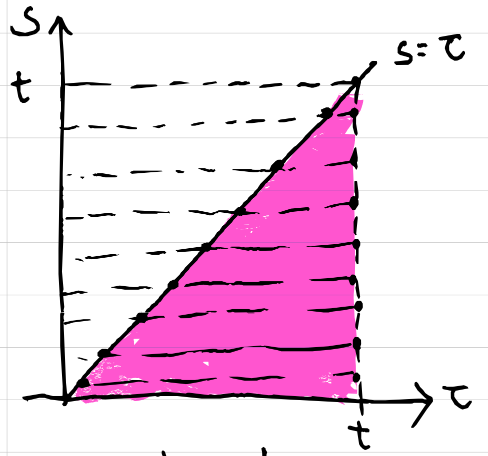
$s = \tau$ since we need to integrate on τ

This is integrating from 0 to t on τ and 0 to τ on s

↳ the given integral

Changing order of integrals:





This is integrating from 0 to S on S and $\tau = s$ to t on τ

So integral becomes

$$\int_0^t \dot{x}(\tau)\,d\tau = \int_0^t \dot{x}(0)\,d\tau + \frac{1}{m}\int_0^t\int_s^t F(s)\,d\tau\,ds$$

$$\Rightarrow x(t) - x(0) = \dot{x}(0)t - \dot{x}(0)\cdot 0 + \frac{1}{m}\int_0^t\left(\int_s^t d\tau\right)F(s)\,ds$$

$$\Rightarrow \boxed{x(t) = x_0 + v_0 t + \frac{1}{m}\int_0^s (t-s)F(s)\,ds}$$

↳ general solution

Example problem 2 (Oscillating Force)

Let $F = A\sin(wt)$

A, w are constants $A, w \in \mathbb{R}$

Initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$

Find $x(t)$, $v(t) = \dot{x}(t)$

Solution:

So we have

$$m\ddot{x} = A\sin(wt)$$

$$\Rightarrow \ddot{x} = \frac{1}{m}A\sin(wt)$$

$$\Rightarrow \int_0^t \ddot{x}(s)\,ds = \int_0^t \frac{1}{m}A\sin(wt)\,dt$$

$$\Rightarrow \dot{x}(t) - \dot{x}(0) = \frac{1}{m}\left[-\frac{A}{w}\cos(wt)\right]_0^t$$

$$\Rightarrow \dot{x}(t) = v_0 + \frac{A}{mw}\left[1 - \cos(wt)\right] = v(t)$$

Integrating again

$$\int_0^t \dot{x}(s)\,ds = \int_0^t v_0 + \frac{A}{mw}\left[1 - \cos(ws)\right]ds$$

$$\Rightarrow x(t) - x(0) = \int_0^t v_0 + \frac{A}{mw} - \frac{A}{mw}\cos(ws)\,ds$$

$$\Rightarrow x(t) = x_0 + v_0 t + \frac{At}{mw} - \frac{A}{mw^2}\sin(ws)$$

$$\Rightarrow x(t) = x_0 + t\left[v_0 + \frac{At}{mw}\right] - \frac{A}{mw^2}\sin(ws)$$

is the general solution for $x(t)$

It is interesting that although the force is an oscillating function of time with 0 mean (over a period $T = 2\pi/w$), the solution is not necessarily periodic and has a constant linearly growing term

3.3) Motion with a force depending on velocity

When a body moves in a liquid or gaseous medium, that medium affects the motion

The interaction of the body and the medium results in a friction (or drag) force exerted on the body by the medium.

The drag force acting on the body depends on the properties of the medium through which the body is travelling, the shape of the body and its velocity.

Defn: Stokes Drag:

It is known from experiments that when the velocity of the body is small or size of the body is small, the magnitude of resistance is proportional to magnitude of the velocity i.e.

$$\boxed{\vec{F} = -\Gamma \vec{v} = -\Gamma \dot{\vec{x}}}$$

where Γ is the drag (resistance or stokes friction) coefficient.

Usually this force is called Stokes drag or stokes friction.

It is also known for sufficiently large v, the drag force can be proportional to the square of the velocity, i.e.

$$\boxed{\vec{F} = -k|\vec{v}|\vec{v}}$$

where k is a constant coefficient.

Example problem 3: (Motion under Stokes drag force)

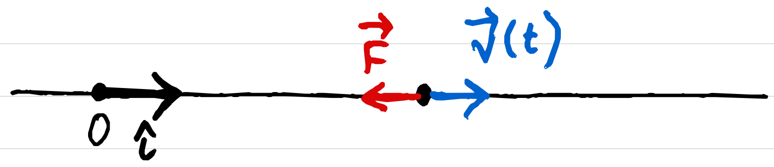
Consider a body of mass m moving in in air and assume that its position and velocity at time $t=0$ are x_0, v_0, i.e.

$$x(0)=x_0 \text{ and } \dot{x}(0)=v_0$$

No external forces are acting on the body for $t>0$ so that it is moving only under the action of Stokes friction.

(i) Find motion of the body for $t>0$, i.e. find $x(t)$ for $t>0$

Solution:



We shall use the co-ordinate axis whose positive direction coincides with the direction of initial velocity. Drag force is given by the defn on previous page.

So equation of motion is

$$m\ddot{\vec{x}} = -\Gamma\dot{\vec{x}}$$

or equivalently the scalar equation

$$m\ddot{x} = -\Gamma\dot{x}$$

Solving ODE, using missing dependant variable method

$$m\ddot{x} = -\Gamma\dot{x} \Rightarrow \ddot{x} + \frac{\Gamma}{m}\dot{x} = 0$$

Let $v=\dot{x} \Rightarrow \dot{v}=\ddot{x}$

So ODE becomes

$$\dot{v} + \frac{\Gamma}{m}v = 0 \Rightarrow \int \frac{1}{v}\frac{dv}{\cancel{dt}}\cancel{dt} = \int \frac{-\Gamma}{m}dt$$

$$\Rightarrow \ln v = -\frac{\Gamma}{m}t + C$$

$$\Rightarrow v(t) = e^{-\frac{\Gamma}{m}t + C}$$

$$\Rightarrow v(t) = e^{C}e^{-\Gamma/m\, t}$$

Using initial conditions $\dot{x}(0)=v(0)=v_0$

$$v(0) = v_0 = e^{C}.e^{-\Gamma/m\, 0} \Rightarrow e^{C} = v_0$$

So

$$v(t) = v_0 e^{-\Gamma/m\, t}$$

Integrating again

$$v(t) = v_0 e^{-(\Gamma/m)t} \Rightarrow \dot{x}(t) = v_0 e^{-(\Gamma/m)t}$$

$$\Rightarrow \int \frac{dx}{dt}\, dt = \int v_0 e^{-(\Gamma/m)t}\, dt$$

$$\Rightarrow x(t) = -\frac{m}{\Gamma} v_0 e^{-\frac{\Gamma}{m}t} + C$$

Using boundary conditions

$$x(0) = x_0 \Rightarrow x_0 = -\frac{m}{\Gamma} v_0$$

$$\Rightarrow x_0 + \frac{m}{\Gamma} v_0 = C$$

So

$$x(t) = x_0 + \frac{m}{\Gamma} v_0 \left(1 - e^{-\frac{\Gamma}{m}t}\right)$$

is the general solution.

Note that $x(t) \to x_0 + \frac{m}{\Gamma} v_0$ as $t \to \infty$.

so that the <u>distance travelled</u> by the body for the <u>infinite time interval</u> <u>$(0, \infty)$</u> is <u>finite</u> and equal to

$$x(\infty) - x_0 = v_0 \frac{m}{\Gamma}$$

b) Check whether your answer is dimensionally correct.

Solution: First determine physical dimensions of Γ. It follows from the equation of motion that two physical quantities can only be equal if and only if they have the same physical dimensions that

$$[m][\ddot{x}] = [\Gamma][\dot{x}] \Rightarrow \frac{M\cancel{L}}{T^{\cancel{2}}} = [\Gamma]\frac{\cancel{L}}{\cancel{T}}$$

$$\Rightarrow [\Gamma] = \frac{M}{T}$$

Our answer for $x(t)$ only makes sense if

$$[x(t)] = \left[\frac{m v_0}{\Gamma}\right] = L \qquad (*1)$$

$$\left[\frac{\Gamma}{m} t\right] = 1 \qquad (*2)$$

verifying $(*1)$

$$\left[\frac{m v_0}{\Gamma}\right] = \frac{\cancel{M} \cdot L/\cancel{T}}{\cancel{M}/\cancel{T}} = L$$

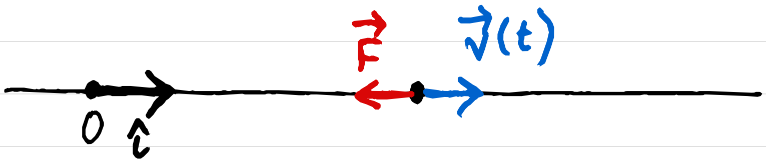
$$\left[\frac{\Gamma}{m} t\right] = \frac{\cancel{M}/T \cdot T}{\cancel{M}} = \frac{1}{T} \cdot T = 1$$

Therefore our formula for $x(t)$ is dimensionally correct.

Example problem 4: (Motion under a friction force which is quadratic in velocity)

Consider same problem as problem 3

Solution:
(i)



Initial conditions are

$$\dot{x}(0) = v_0, \qquad x(0) = x_0$$

Equation of motion is

$$m\ddot{\vec{x}} = -K|\dot{\vec{x}}|\dot{\vec{x}} \Rightarrow m\ddot{x}\hat{i} = -K|\dot{x}|\dot{x}\hat{i}$$

which gives scalar equation

$$m\ddot{x} = -K|\dot{x}|\dot{x}$$

Now we assume velocity does not change sign for $t>0$, (this can be verified once we know $v(t)$):

$v(t)>0$ for $t \in [0,\infty)$.

With this assumption, $|\dot{x}| = \dot{x}$ and eqn of motion can be rewritten as

$$m\ddot{x} = -K\dot{x}^2$$

Solving ODE using missing dependant variable,

Let $v = \dot{x} \Rightarrow \dot{v} = \ddot{x}$

$$m\ddot{x} = -K\dot{x} \Rightarrow m\dot{v} = -Kv^2$$

$$\Rightarrow m\dot{v} + Kv^2 = 0$$

is the linear homogeneous ODE

Solving ODE

$$m\dot{v} + kv^2 = 0 \Rightarrow \dot{v} = -\frac{k}{m}v^2$$

$$\Rightarrow \int \frac{1}{v^2} \cdot \frac{dv}{dt}\,dt = \int -\frac{k}{M}\,dt$$

$$\Rightarrow \frac{-1}{v} = -\frac{kt}{m} - c$$

$$\Rightarrow \frac{1}{v} = \frac{kt}{M} + c$$

$$\Rightarrow v(t) = \frac{1}{\frac{kt}{m} + c}.$$

let $q = k/m$. Then

$$v(t) = \frac{1}{qt + c}$$

Using initial conditions $v(0) = \dot{x}(0) = v_0$

$$v(0) = v_0 = \frac{1}{c} \Rightarrow c = \frac{1}{v_0}$$

So $$v(t) = \frac{1}{qt + \frac{1}{v_0}}$$

$$\Rightarrow v(t) = \frac{v_0}{qv_0t + 1}$$

Note that velocity is a decreasing function of t for all $t > 0$.

And $v(t) \to 0$ as $t \to \infty$

So body moves slower and slower with time and stops in the limit $t \to \infty$.

Also note $v(t) > 0 \ \forall\, t > 0$ so our assumption holds.

Further we have

$$v(t) = \frac{v_0}{1+v_0qt} \Rightarrow \dot{x} = \frac{v_0}{1+v_0qt}$$

$$\Rightarrow \int \frac{dx}{dt} dt = \int \frac{v_0}{1+v_0qt} dt$$

$$\Rightarrow x = v_0 \cdot \frac{1}{v_0 q} \ln|1+v_0qt| + c$$

$$\Rightarrow x(t) = \frac{1}{q} \ln|1+v_0qt| + c$$

Using initial conditions

$$x(0) = x_0 \Rightarrow x(0) = \frac{1}{q} \ln|1| + c$$

$$\Rightarrow x_0 = c$$

So

$$x(t) = x_0 + \frac{1}{q} \ln|1+v_0qt|$$

is the general solution.

Note that in example problem 4,

$$x(t) \to \infty \text{ as } t \to \infty$$

so that the distance travelled by the body for the infinite time interval $(0, \infty)$ is infinite.

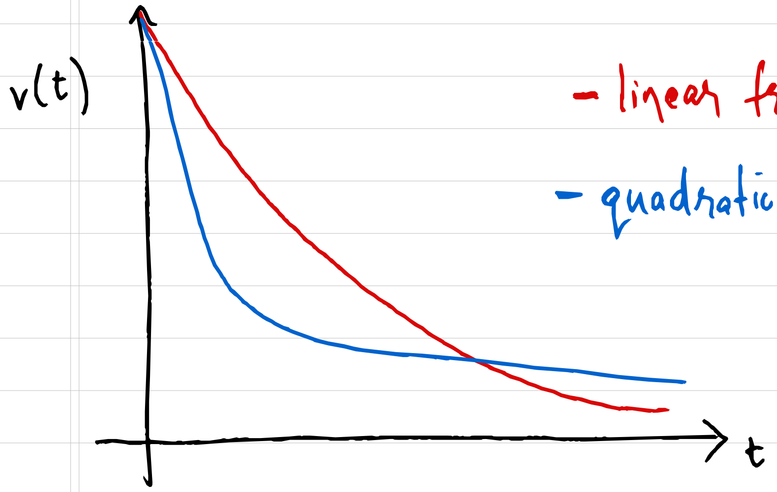
Thus we have obtained a paradoxical result:

For large v, the magnitude of the stokes friction force given by $\vec{F} = -\Gamma\vec{v}$ is smaller than the magnitude given by $\vec{F} = -k|\vec{v}|\vec{v}$ but our result shows that in the former case, the distance travelled by the body is finite and in the latter case, it is infinite.

The explanation for this paradox is that the formula $\vec{F} = -k|\vec{v}|\vec{v}$ is only valid for sufficiently large velocity.

when the velocity becomes small and it always does, the formula becomes incorrect.

(in ex problem) So formulas for $v(t)$ and $x(t)$ are only valid and describe the motion of the body only on some finite interval 0 to t_1 for some t_1.



From the above graphs, one can essentially see that

- initially when the velocity of the body is high, the quadratic friction is more efficient at decelariting the body
- when the velocity becomes sufficiently small, linear friction is more efficient.

(b) As in example 3, we need to find physical dimensions of k.

From equation of motion we have

$$[m\ddot{x}] = [k\dot{x}^2] \Rightarrow [m][\ddot{x}] = [k][\dot{x}^2]$$

$$\Rightarrow M.\frac{\cancel{L}}{\cancel{T^2}} = [k].\frac{L^{\cancel{2}}}{\cancel{T^2}}$$

$$\Rightarrow [k] = \frac{M}{L}$$

Formula for $x(t)$ only makes sense if

(*1) $\left[\frac{1}{q}\right] = L$ and $[v_0 q t] = 1$ (*2)

($v_0 q t$ is dimensionless since log can only accept dimensionless values)

Verifying (*1)

$$q = \frac{k}{m} \Rightarrow \frac{1}{q} = \frac{m}{k} \Rightarrow \left[\frac{1}{q}\right] = \left[\frac{m}{k}\right]$$

$$\Rightarrow \left[\frac{1}{q}\right] = [m].\frac{1}{[k]}$$

$$\Rightarrow \left[\frac{1}{q}\right] = \cancel{M}.\frac{1}{\cancel{M}/L} = L$$

$$\left[\frac{1}{q}\right] = L \Rightarrow [q] = \frac{1}{L}$$

$$[v_0 q t] = [v_0][q][t]$$

$$= \frac{\cancel{L}}{\cancel{T}}.\frac{1}{\cancel{L}}.\cancel{T} = 1 \Rightarrow [v_0 q t] = 1$$

We conclude that our formula for $x(t)$ is dimensionally correct.

3.4) Motion under a force depending on position

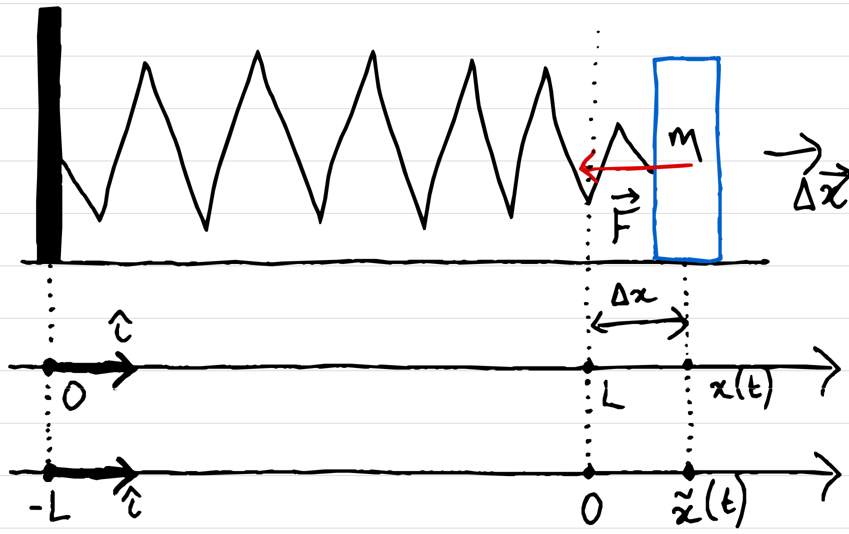
Consider a particle of mass m moving on a straight line under the action of a force F which depends only on the co-ordinate x of the particle.

In this case the equation of motion has the form

$$\vec{F} = F(x)\hat{\imath} \Rightarrow \boxed{\begin{array}{l} m\ddot{x}\hat{\imath} = F(x)\hat{\imath} \\ \Rightarrow m\ddot{x} = F(x) \end{array}}$$

<u>Example problem 5</u>: (motion of a body attached to a spring):

Consider the system shown in fig below:



A body attached to right end of spring of natural length L.
Left end of spring is attached to a wall that is <u>fixed</u>
Body lies on a smooth surface and is free to move on this surface without friction.
<u>Equation of motion</u> is given by

$$m\ddot{x} = -k(x-L)$$

where k is the elastic constant of spring,
L is the natural length of L,

Find motion of the body for $t>0$ given that

$$x(0) = x_0, \quad v(0) = \dot{x}(0) = v_0$$

Solution:

$$m\ddot{x} = -k(x-L) \Rightarrow \ddot{x} = -\frac{k}{m}(x-L) \quad \text{(dividing by } m\text{)}$$

Let $\frac{k}{m} = w^2$

$$\ddot{x} = -\frac{k}{m}(x-L) \Rightarrow \ddot{x} = -w^2(x-L)$$

$$\Rightarrow \ddot{x} = -w^2 x + w^2 L$$

$$\Rightarrow \ddot{x} + w^2 x = w^2 L$$

This is a linear inhomogeneous second order ODE with constant coefficients

General soln is sum of particular solution $x_p(t)$ and general solution $x_h(t)$ of homogeneous eqn:

$$x(t) = x_p(t) + x_h(t)$$

Finding solution of homogeneous eqn:

$$\ddot{x}_h + w^2 x_h$$

Assume solution of the form $x = e^{\lambda t}$ (ansatz) giving auxillary eqn

$$\lambda^2 + w^2 = 0 \Rightarrow \lambda = \pm i w$$

So general soln to homogeneous eqn is

$$x_h(t) = c_1 \sin wt + c_2 \cos wt$$

$c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Finding particular solution

ansatz: $x_p(t) = \alpha t + \beta$

$x_p' = \alpha$, $x_p'' = 0$

Substituting

$0 + w^2(\alpha t + \beta) = w^2 L$

$\Rightarrow w^2 \alpha t + w^2 \beta = w^2 L \Rightarrow \alpha = 0$ and

$\beta = L. \Rightarrow x_p(t) = L$

So the general soln is

$x(t) = L + c_1 \sin wt + c_2 \cos wt$

Substituting initial | boundary conditions

$x(0) = x_0 = L + c_2 \Rightarrow c_2 = x_0 - L$

$\dot{x}(0) = w c_1 \cos(w0) - w c_2 \sin(w0)$

$\Rightarrow c_1 = \frac{v_0}{w}$

Hence $x(t) = L + \frac{v_0}{w} \sin(wt) + (x_0 - L)\cos(wt)$

Discovering energy:

Let $z(t) = x(t) - L$

$$z_0 = x_0 - L$$

Then $x(t) = L + \frac{v_0}{\omega}\sin(\omega t) + (x_0 - L)\cos(\omega t)$

can be rewritten as

$$z(t) = \frac{v_0}{\omega}\sin(\omega t) + z_0\cos(\omega t)$$

This is a solution of the equation

$$\ddot{z} + \omega^2 z = 0$$

satisfying initial conditions $z(0) = z_0$, $\dot{z}(0) = v_0$

Equation $\ddot{z} + \omega^2 z = 0$ can be interpreted as the equation of motion of a body attached to a spring relative to the co-ordinate axis whose origin is at equilibrium position of the body.
(where length of the spring is equal to its natural length).

Differentiang $z(t)$ we get

$$\dot{z}(t) = v_0 \cos(wt) - z_0 \sin(wt)$$

<u>The idea:</u> lets eliminate the explicit dependance of t and see what the relation b/w $z(t)$ and $\dot{z}(t)$ looks like.

$$z(t) = \frac{v_0}{w}\sin(wt) + z_0\cos(wt)$$

$\Rightarrow$

$$z^2(t) = \left(\frac{v_0}{w}\sin(wt) + z_0\cos(wt)\right)^2$$

$\Rightarrow$

$$z^2(t) = \frac{v_0^2}{w^2}\sin^2(wt) + z_0^2\cos^2(wt) + \frac{2v_0}{w}\sin(wt)\cos(wt)$$

Dividing by z_0^2

$$(*1)\quad \frac{z^2(t)}{z_0^2} = \left(\frac{v_0}{z_0 w}\right)\sin^2(wt) + \cos^2(wt) + \frac{2v_0}{z_0 w}\sin(wt)\cos(wt)$$

Similarly for $\dot{z}(t)$

$$\dot{z}(t) = v_0\cos(wt) - z_0 w\sin(wt)$$

$\Rightarrow$

$$\dot{z}^2(t) = \left(v_0\cos(wt) - z_0 w\sin(wt)\right)^2$$

$\Rightarrow$

$$\dot{z}^2(t) = v_0^2\cos^2(wt) + z_0^2 w^2\sin^2(wt) - v_0 z_0 w\cos(wt)\sin(wt)$$

Dividing by $z_0^2 w^2$

$$(*2)\quad \dot{z}^2(t) = \left(\frac{v_0}{z_0 w^2}\right)^2\cos(wt) + \sin^2(wt) - 2\frac{v_0}{z_0 w}\sin(wt)\cos(wt)$$

Adding (*1) and (*2)

$$\frac{z^2(t)}{z_0^2} + \frac{\dot{z}^2(t)}{z_0^2 w^2} = \frac{v_0^2}{z_0^2 w} + 1$$

$\Rightarrow$ (multiplying by $z_0^2 w^2$)

$$\dot{z}^2(t) + w^2 z^2(t) = v_0^2 + w^2 z_0^2$$

$\Rightarrow$ (multiplying by $m/2$)

$$\frac{m}{2}\dot{z}(t) + \frac{k}{2} z^2(t) = \frac{m}{2} v_0^2 \quad \frac{m}{2} w^2 z_0^2$$

$\Rightarrow$ (substituting $w^2 = k/m$ from ex 5)

$$\frac{m}{2}\dot{z}(t) + \frac{k}{2} z^2(t) = \frac{m}{2} v_0^2 + \frac{k}{2} z_0^2 \qquad \text{eq}(*1)$$

Let

$$E(t) = \frac{m}{2}\dot{z}(t) + \frac{k}{2} z^2(t)$$

Eq (*1) says quantity $E(t)$ remains same as initial value no matter what $t > 0$

i.e.

$$E(t) = \frac{m v_0^2}{2} + \frac{k}{2} z_0^2$$

Defn: Energy:

Physical property that have the property explained previously are called <u>constants of motion</u>, and $E(t)$ given by

$$E(t) = \frac{m\dot{z}^2(t)}{2} + \frac{kz^2(t)}{2} \qquad (1)$$

is called <u>energy</u>.

The fact energy does not change is known as <u>law of conservation of energy</u>.

The <u>first term</u> in (1) $m\dot{z}^2/2$ is called <u>kinetic energy</u> (of a body of mass m moving with velocity v)

The <u>second term</u> in (1) $kz^2/2$ is called the <u>potential energy</u>

In each particular physical problem, potential energy has its own nature.

3.5) Motion in a potential

In preceeding section we have found a solution of the equation of motion given by

$$m\ddot{x} = -kx$$

that describes motion of a body of mass m attached to a spring (with elastic constant k.) Then we used this solution to show the energy E of the system

$$E = \frac{m\dot{x}^2(t)}{2} + \frac{kx^2(t)}{2} \qquad (2)$$

does not depend on time i.e.

$$E(t) = E(0).$$

Can we deduce that the energy is conserved without having to solve it?

↳ Answer is yes.

Let equation of motion by

$$m\ddot{x} = -kx$$

multiplying through by $\dot{x}$

$$m\ddot{x}\dot{x} = -kx\dot{x} \Rightarrow m\ddot{x}\dot{x} + kx\dot{x} = 0$$

We observe that

$$\dot{x}\ddot{x} = \dot{x}\frac{d\dot{x}}{dt} = \frac{d}{d\dot{x}}\left(\frac{\dot{x}^2}{2}\right)\cdot\frac{d\dot{x}}{dt}$$

$$= \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right) \quad \left[\text{by chain rule}\right]$$

$$x\dot{x} = x\frac{dx}{dt} = \frac{d}{dx}\left(\frac{x^2}{2}\right)\cdot\frac{dx}{dt}$$

$$= \frac{d}{dt}\left(\frac{x^2}{2}\right) \quad \left[\text{by chain rule}\right]$$

Therefore

$$m\ddot{x}\dot{x} + k\dot{x}x = 0 \Rightarrow \frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + \frac{kx^2}{2}\right)$$

$$\Rightarrow \frac{dE}{dt} = 0$$

with E given by (2) on previous page.

Question:
Is the law of conservation of energy generic or is it just a random coincidence that the energy of a body + spring is conserved

Answer: To answer this question, let look at the general equation of motion of a particle of mass m under a position dependant force:

$$m\ddot{x} = F(x)$$

Again multiplying by $\dot{x}$ yields

$$m\dot{x}\ddot{x} = F(x)\dot{x} \Rightarrow \frac{d}{dt}\left(m\frac{\dot{x}^2}{2}\right) = F(x)\dot{x}$$

Now let $V(x)$ be defined by the equation

$$\boxed{F(x) = -\frac{dV(x)}{dx}}$$

Alternatively we can define $V(x)$ as

$$V(x) = -\int F(x)\,dx \quad \text{or} \quad V(x) := -\int_{x_0}^{x} F(s)\,ds$$

For arbitrarily chosen x_0.

Then we have

$$F(\dot{x})\,\dot{x} = -\frac{dV}{dx}\dot{x} = -\frac{d}{dt}V(x) \qquad \left[\text{by chain rule}\right]$$

Therefore we obtain

$$\frac{d}{dt}\left(\frac{m\dot{x}^2}{2} + V(x)\right) = 0$$

or

$$\frac{d}{dt}E = 0$$

where

$$\boxed{E = \frac{m\dot{x}^2}{2} + V(x)} \qquad (3)$$

Defn: The potential and conservation of energy

The quantity $\frac{1}{2}m\dot{x}^2$ is the kinetic energy of the particle.

$V(x)$ is by definition the potential energy of the particle, or the potential of the force $F(x)$ defined by

$$\boxed{\frac{dV}{dx} = -F(x)}$$

E is called the energy (or the total energy)

Equation

$$\frac{dE}{dt} = 0$$

expresses conservation of energy of the particle where

$$E = \frac{m\dot{x}^2}{2} + V(x)$$

Note that if $V(x)$ is a potential for $F(x)$, i.e. $V(x)$ satisfies

$$F(x) = -\frac{dV}{dx}$$

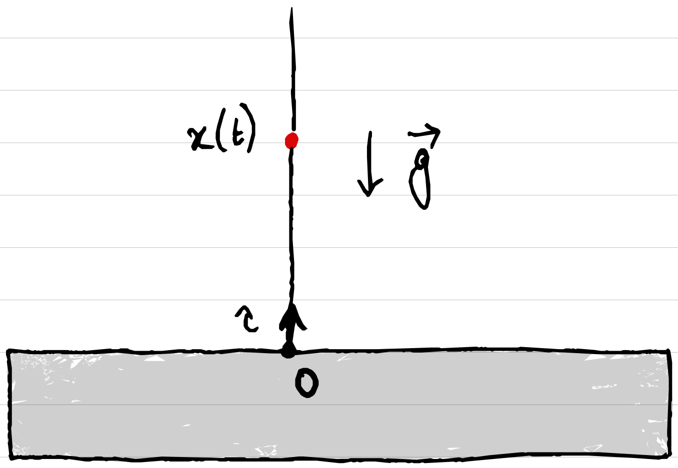
then $\tilde{V}(x) = V(x) + C$ is also a potential for $F(x)$ for any arbitrary constant C.

This means potential energy $V(x)$ is defined up to an arbitrary constant.

Examples of potentials :

• Uniform gravity:

If x axis vertical and directed upwards,



$$\underline{F} = m\underline{g} \Rightarrow F(x)\underline{i} = -mg\underline{i}$$

$$\Rightarrow F(x) = -mg \quad \text{(scalar eqn)}$$

$$V(x) = -\int F(x)\,dx$$

$$= -\int -mg\,dx$$

$$= mg\int dx$$

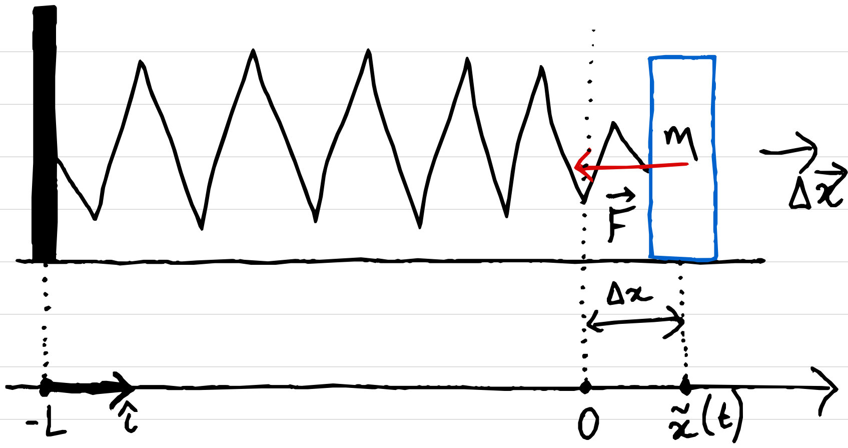
$$\Rightarrow V(x) = mgx + C$$

Convenient to choose $C=0$, Then

$$\boxed{V(x) = mgx}$$

So the potential energy of a body (equivalently potential of uniform gravity force) is 0 when it is on the ground.

- A body + spring



In this case

$$F = -kx$$

Therefore

$$V(x) = -\int F(x)\,dx$$

$$= k\int x\,dx$$

$$= \frac{kx^2}{2} + C$$

For any arbitrary constant C.

Again it is convenient to choose $C=0$. Then

$$\boxed{V(x) = \frac{kx^2}{2}}$$

and the elastic energy of the spring or equivalently the potential energy of the body or potential of force $F=-kx$ is 0 when length of spring is equal to its natural length.

• Newtownian gravity:

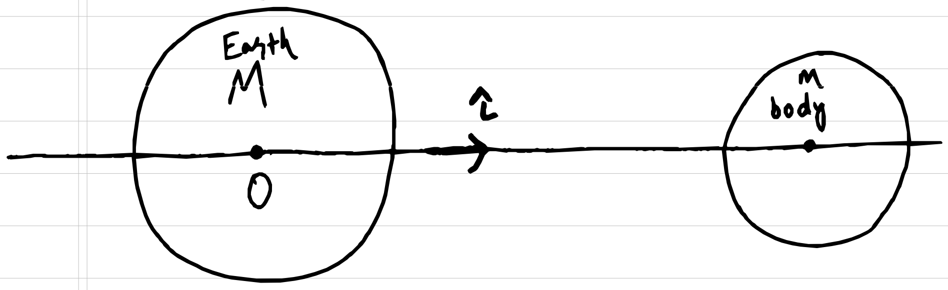
The newtown gravitatational force of attraction between 2 bodies of mass m and mass M is

$$\boxed{F = \frac{GmM}{r^2}}$$

where G is the gravitational constant and r is the distance between the masses.

Let:

- M be the mass of the earth.
- m be the mass of another spherically symmetric body.
- Introduce x axis that passes through centere of Earth and is directed from Earth to the body



The gravitational force acting on the body is given by

$$F = \frac{GmM}{x^2}$$

Hence

$$v(x) = -\int F(x)\,dx$$

$$= -\frac{Gmm}{x} + C$$

If we choose $C = 0$, then

$$v(x) = -\frac{GmM}{x}$$

This choice corresponds to 0 potential energy of the body when it is far away (at infinity) from the earth.

Note also that $v(x)$ is negative for any $x > 0$.

There is nothing wrong with potential energy being negative.

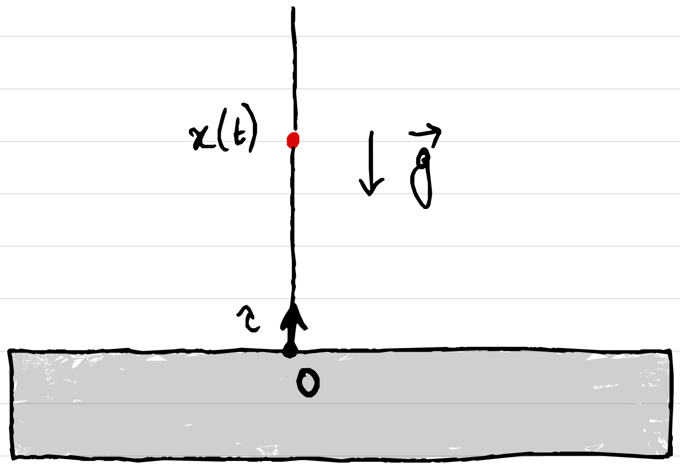
The kinetic energy cannot be negative

The potential energy as well as total energy can be negative

Example problem 6: (Motion under uniform gravity):

Consider vertical motion of a particle of mass m, under the action of the uniform gravity force.

Suppose that initially particle is at position x_0 and has velocity $v_0 > 0$. (The co-ordinate x axis is vertical and directed upwards).
We can use the energy conservation to find for example the highest point of the trajectory of the particle.



We have

$$E\big|_{t=0} = \frac{mv_0^2}{2} + mgx_0$$

Since energy is a constant of motion

$$E(t) = E(0)$$

i.e.

$$\frac{mv^2(t)}{2} + mgx(t) = \frac{mv_0^2}{2} + mgx_0$$

At highest point in trajectory at $t = t^*$
The velocity of particle is 0

$$v(t^*) = 0$$

Therefore

$$E(t^*) = E(0) \Rightarrow mgx(t^*) = \frac{mv_0^2}{2} + mgx_0$$

$$\Rightarrow x(t^*) = x_0 + \frac{v_0^2}{2g}$$

3.6) Using energy conservation to solve equations of motion.

The energy given by

$$E = \frac{m\dot{x}^2}{2} + V(x)$$

is a constant and its value can be found from initial conditions.
Then it follows from the equation for E that

$$\boxed{\dot{x} = \pm\sqrt{2(E-V(x))/m}}$$

$\Rightarrow$

$$\frac{dx}{\sqrt{2(E-V(x))/m}} = \pm dt$$

$$\Rightarrow \boxed{\int \frac{dx}{\sqrt{2(E-V(x))/m}} = \pm t}$$

In principle if we know $V(x)$ (potential), we can evaluate the integral.

This introduces a constant of integration, C and the resulting the equation can be inverted to give the position of the particle as a function of time $x(t)$.

The two constants of integration E and C will appear in this function which will describe all possible motions of the particle consistent with a given force $F(x)$.

Example problem 7: (a body attached to a spring):

Consider a particle of mass m moving under the action force

$$F(x) = -kx.$$

Equation of motion is

$$m\ddot{x} = -kx$$

The general solution is

$$x(t) = c_1 \sin(wt) + c_2 \cos(wt). \quad (*1)$$

$w = \sqrt{\frac{k}{m}}$, c_1 and c_2 are arbitrary constants.

But we can use the method described previously to obtain a solution.

$$V(x) = \frac{kx^2}{2} \Rightarrow E = \frac{m\dot{x}^2}{2} + kx^2$$

$$\Rightarrow \dot{x} = \pm\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}$$

$$\Rightarrow t + C = \pm\int \frac{dx}{\sqrt{\frac{2E}{m} - \frac{k}{m}x^2}}$$

Let $y = x\sqrt{\frac{k}{2E}}$ Then $\sqrt{\frac{2E}{k}}\,dy = dx$

$$t + C = \pm\int \frac{dx}{\sqrt{\frac{2E}{m} - \frac{kx^2}{m}}} \Rightarrow t + C = \pm\int \frac{dx}{\sqrt{\frac{2E}{m}\left(1 - \frac{k}{2E}x^2\right)}}$$

$$\Rightarrow t + C = \pm\sqrt{\frac{m}{2E}}\int \frac{dx}{\sqrt{1 - \frac{k}{2E}x^2}}$$

$$\Rightarrow t + C = \pm\sqrt{\frac{m}{\cancel{2E}}}\sqrt{\frac{\cancel{2E}}{k}}\int \frac{dy}{\sqrt{1-y^2}}$$

$$\Rightarrow t + C = \pm\sqrt{\frac{m}{k}}\int \frac{dy}{\sqrt{1-y^2}}$$

$$\Rightarrow t + C = \pm\sqrt{\frac{m}{k}}\arcsin(y)$$

$$\Rightarrow \pm t + C = \sqrt{\frac{m}{k}}\arcsin\left(x\sqrt{\frac{k}{2E}}\right)$$

$$\Rightarrow \pm\sqrt{\frac{k}{m}}(t + C) = \arcsin\left(x\sqrt{\frac{k}{2E}}\right)$$

$$\Rightarrow \sin\left(\pm\sqrt{\frac{k}{m}}(t + C)\right) = x\sqrt{\frac{k}{2E}}$$

$$\Rightarrow \pm\sin\left(\sqrt{\frac{k}{m}}(t + C)\right) = x\sqrt{\frac{k}{2E}}$$

$$\Rightarrow x(t) = \pm\sqrt{\frac{2E}{k}}\sin\left(\sqrt{\frac{m}{k}}(t + C)\right)$$

So we get a general soln:

$$x(t) = \pm\sqrt{\frac{2E}{K}}\sin\left(\sqrt{\frac{K}{m}}(t+c)\right)$$

or

$$x(t) = A\sin\left(\sqrt{\frac{K}{m}}t + \phi_0\right) \qquad (*2)$$

where A and ϕ_0 are arbitrary constants.

Equations $(*1)$ and $(*2)$ are both general solutions to $m\ddot{x} = -kx$.

Equation $(*2)$ can be written in form $(*1)$ by using

$$\sin(x+y) = \sin(x)\cos y + \cos x\sin(y)$$

3.7) Using energy to describe motion qualitatively

3.7.1 Qualitative analysis of motion:

Consider a particle of mass m moving on a straight line, under the action of a force F with potential $V(x)$.
It turns out that many features of motion can be deduced from the graph of potential energy $V(x)$.

Recall

$$\frac{dV(x)}{dx} = -F(x)$$

so force is the minus slope of the graph $V(x)$

Observations

- If $V'(x_0) > 0$ then the force acting on the particle at point x_0 is negative, i.e. direction of force is opposite relative to the x axis.

- If $V'(x_0) < 0$ then the force acting on the particle at point x_0 is positive, i.e. direction of force is in the same direction as that of the x axis

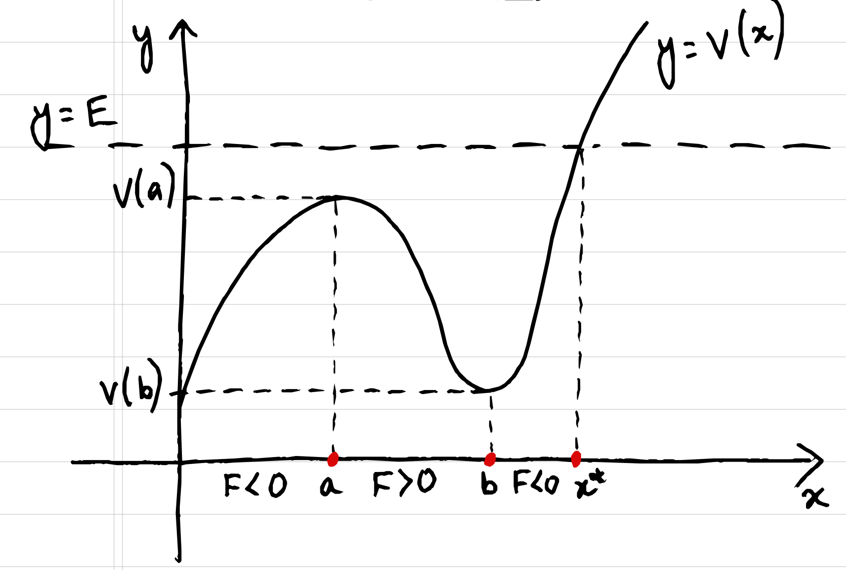
- Since $E = \frac{m\dot{x}^2}{2} + V(x)$ is a constant of motion (whose value is determined by initials conditions for $x, \dot{x}$), we conclude that $V(x) \leq E$. That means the particle whose total energy is E cannot be located at points x such that $V(x) > E$.
  (i.e. points of x for which graph of $y = V(x)$ is above the line $y = E$).

Suppose that graph of potential energy has a local maximum at $x=a$ and a local minimum at $x=b$.

Total energy E of the particle is represented by dashed horizontal lines.

Let x_0, v_0 be initial position of particle and initial velocity of the particle.

• Case 1: $E > V(a)$



According to this graph and our observations, particle cannot be at position $x > x^*$

(a) if $v_0 > 0$: the particle will move in the positive x direction until velocity $v(t) = \dot{x}(t)$ becomes 0

- for particle in $x < a$, the force acts in opposite direction to velocity, decelerating
- for $a < x < b$, the force acts in same direction to velocity hence body is decelerating
- for $b < x < x^*$, the force acts in opposite direction to velocity, hence body is decelerating.

• For $x < x^*$, velocity is non-zero because

$$v = \dot{x} = \sqrt{2(E - V(x))/m}$$

and $E > V(x)$ $\forall x < x^*$

• At $x = x^*$, velocity of the particle is 0 because

$$v = \dot{x} = \sqrt{2(E - V(x))/m}$$

and $E = V(x)$ for $x = x^*$

The force at point $x = x^*$ is nonzero and negative; $F(x^*) = -V'(x^*)$, so particle will start moving in the negative direction of the x axis and will keep moving to ∞ (infinity).

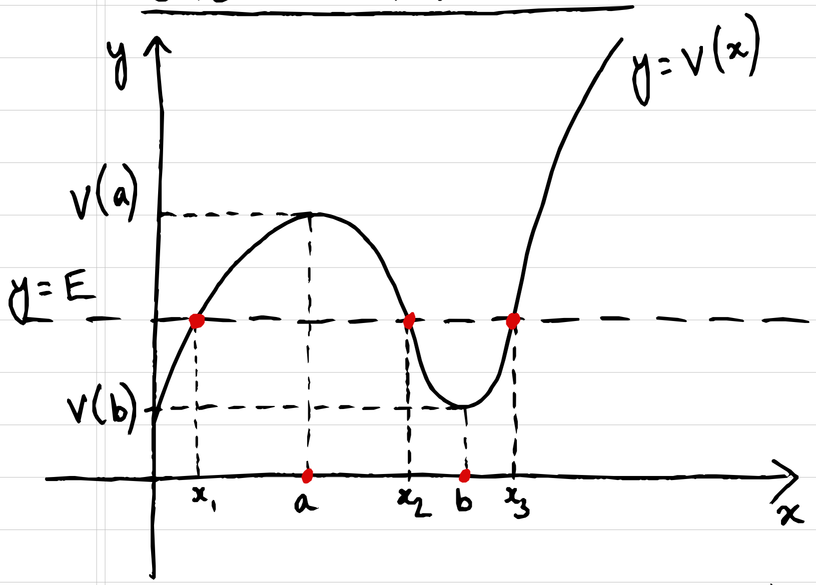
↳ we will say "particle escapes to ∞"

Defn: Points at which velocity changes direction to the opposite are called turning points of motion.

(example in case 1, $x = x^*$ is a turning point)

(b) if $v_0 < 0$: the particle will continue to move in the negative direction of the x-axis, and eventually escape to infinity

• Case 2: $V(b) < E < V(a)$



According to this graph, the particle cannot be at points $x > x_3$, or at $x \in (x_1, x_2)$.
It can either move in the interval $(-\infty, x_1]$ or in the interval $[x_2, x_3]$.

(a) If $x_0 < x_1$: then depending on direction of initial velocity the particle will either continue to move in the negative direction of the x axis or move towards the turning point x_1 and then back.

↳ in both cases it will escape to ∞

(b) If $x_2 < x_0 < x_3$:

If $v_0 > 0$: the particle will move towards the turning point x_3, where the direction of the velocity is reversed, and then back towards the other turning point x_2.

At x_2, the direction of the velocity changes to the opposite sign and starts moving towards x_3.

And this motion will repeat itself indefinately.
Thus the motion of the particle is periodic!

↳ This is an example of finite motion i.e. for all $t > 0$, motion occurs in a finite interval of the x-axis.

If $v_0 < 0$: the motion is qualitatively the same

For case 2: period of motion can be determined as follows:

First recall that for a given value of E, the velocity can be found using the formula

$$\dot{x} = \pm\sqrt{2(E-V(x))/m} \qquad (*3)$$

Then we consider the motion of the particle from point x_2 to x_3.
We assume that at time t_2, particle is at point x_2 $(x(t_2) = x_2)$
Since x_2 is the turning point of motion, the particles velocity is 0, i.e.

$$\dot{x}(t_2) = 0.$$

The force is positive. So particle will start moving with the velocity given by the + sign of (*3):

$$\dot{x} = \sqrt{2(E-V(x))/m} \qquad (*4).$$

(The plus sign was chosen because the particle is moving from point x_2 to x_3, the velocity is positive).

Suppose that it will arrive arrive at point x_3 at some later time t_3, i.e. $x(t_3) = x_3$.

After it reaches x_3, it will start moving back with the velocity that has exactly the same magnitude but opposite direction and after a while, it will reach x_2.

Then the same motion will repeat again and again. Since the magnitude of the velocity is the same irrespective of whether the particle moves from x_2 to x_3 or x_3 to x_2, it will take the same time for it to move from x_2 to x_3 and x_3 to x_2.

Hence time period T is equal to twice the time needed to move from x_2 to x_3, i.e.

$$T = 2(t_3 - t_2).$$

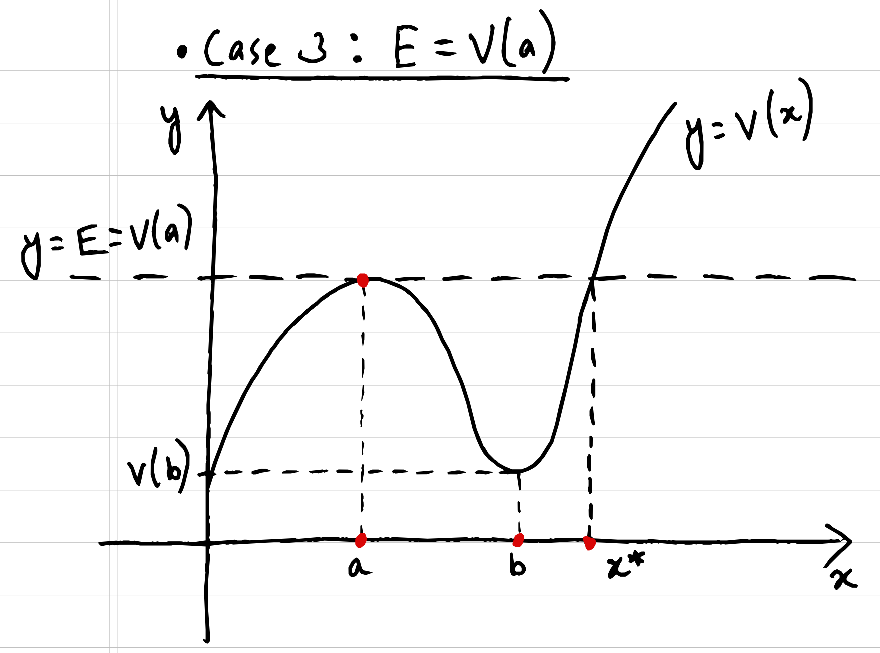
Seperation of variables in eqn (*4) and integration yield the formula for T

$$\int_{t_2}^{t_3} dt = \int_{x_2}^{x_3} \frac{dx}{\sqrt{2(E-V(x))/m}}$$

$\Rightarrow$

$$\boxed{T = 2\int_{x_2}^{x_3} \frac{dx}{\sqrt{2(E-V(x))/m}}}$$

• Case 3 : $E = V(a)$



(a) If $a < x_0 < x^*$: then depending on direction of initial velocity, the particle will continue to move in the negative direction of x-axis ($v_0 < 0$) towards the point of local maximum of V, $x = a$ or move towards turning point x^* ($v_0 > 0$) and then back.

In both cases, it will be approaching the point $x = a$.
Note that $V(x)$ has a local maximum at $x = a$, i.e. $V'(a) = 0$ and force is 0 at $x = a$.
If the particle was at this point, it would have zero acceleration ($F(x) = 0$) and 0 velocity ($E = V$) hence it would remain there forever.

It turns out the particle will never reach this point. It will need infinite time for that.

To show this, we consider motion of the particle near the point $x = a$.

So we assume that $v_0 < 0$ (towards $x = a$) and initially it is close to $x = a$, i.e.

$$x_0 - a \quad \text{is small}$$

For small $x - a$, we expand the potential energy $V(x)$ into Taylor series about the point $x = a$ and retain only the first 3 terms:

$$V(x) \approx V(a) + (x-a)V'(a) + \frac{(x-a)^2}{2} V''(a).$$

Since $x = a$ is the point of local maximum of $V(x)$, it first derivative vanishes at this point $V'(a) = 0$ and $V''(a) < 0$. Hence

$$V(x) \approx V(a) - k\frac{(x-a)^2}{2}$$

where $\kappa \equiv -V''(a) > 0$.

Since particle is moving in negative x direction we have

$$v = -\sqrt{2(E - V(x))/m} \Rightarrow v = -\sqrt{\frac{k}{m}(x-a)^2}$$

$$\Rightarrow v = -\sqrt{\frac{k}{m}}\,(x-a)$$

Solving $v = \dot{x} = -\sqrt{\frac{k}{m}}\,(x-a)$
(linear first order ODE) we obtain

$$\boxed{x(t) = a + (x_0 - a)e^{-\sqrt{\frac{k}{m}}\,t}} \qquad (*5)$$

Evidently, $x(t) > a$ for any finite $t > 0$ and

$$x(t) \to a \text{ as } t \to \infty$$

(b) If $x_0 < a$ and $v_0 < 0$: the particle will continue moving in the negative direction of the x-axis and will escape to infinity.

If $x_0 < a$ and $v_0 > 0$, the particle will move towards $x = a$. When it is sufficiently close to a, its motion is approximately described by (*5)

$$x(t) = a + (x_0 - a)e^{-\sqrt{\frac{k}{m}}\,t}$$

So it will never reach $x = a$

3.7.2 Equilibrium Points and their Stability

The point $x=a$ and $x=b$ in the previous graphs are critical (also called stationary) points of the potential energy $V(x)$

$$\hookrightarrow V'(x) = 0$$

and therefore force acting on the particle at these points is 0.
Hence if initially the particle is at rest at any of these points, it will remain there forever.

Defn: Equilibrium

If $x(0)=a$, $\dot{x}(0)=0$ and $V'(a)=0$ then the equation of motion,

$$m\ddot{x} = -V'(x)$$

has a constant solution $x(t)=a$ for all $t>0$

Such points are called equilibrium position (or simply equilibria) of the particle.

Now let x^* be an equilibrium point of $V(x)$ of a particle in a potential $V(x)$.

Suppose that the initial position is slightly pertubed from its state of rest at this point, i.e.

$$x(0) = x^* + \delta\,,\quad \dot{x}(0) = \varepsilon$$

where δ, ε are small. This particle will start to move and there are 2 possibilities

Defn: Stability

- The pertubed motion of particle will remain close to equilibrium for all $t > 0$
  If this is so for all small pertubations, (i.e. for all small δ and ε), the equilibrium is said to be stable.

- The pertubed particle will move away from the equilibrium.
  If there is atleast one small pertubation (of initial position or velocity) such that the particle moves away from the equilibrium, then this equilibrium is called unstable.

The graphs shown previously has 2 equilibrium points

1) at $x=b$ 2) at $x=a$

near $x=b$: 1) If we slightly perturb the equilibrium of the particle at $x=b$, the particle will oscillate near this equilibrium, i.e, it will remain close to it. To see this suppose:

- The particle is located at point $x>b$. Since $x=b$ is the local minimum of V, we observe:

$$V'(x)>0 \Rightarrow F(x)<0 \quad \text{for } x>b$$

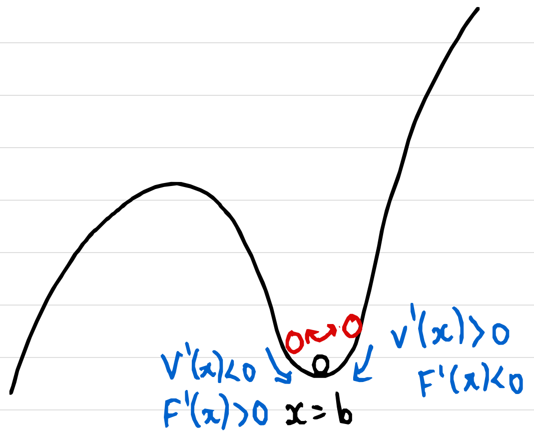
so force is directed towards equilibrium and will move back towards it

- Similarly if initially pertubed particle is located at point $x<b$ then

$$V'(x)<0 \Rightarrow F(x)>0 \quad \text{for } x<b$$

Then so that again force is directed towards the equilibrium and particle will move back towards it.

Thus $x=b$ is a stable equilibrium



Thus equilibrium is stable if it is a local minimum of $V(x)$ i.e.

$$V''(x)<0$$

at the equilibrium

near $x=a$:

2) If we slightly perturb the equilibrium of the particle at $x=a$, the particle will move away from this equilibrium, i.e., it will escape to ∞ or move towards point b. To see this

- The particle is located at point $x>a$. Since $x=a$ is the local maximum of V, we observe:

$$V'(x)<0 \Rightarrow F(x)>0 \quad \text{for } x>a$$

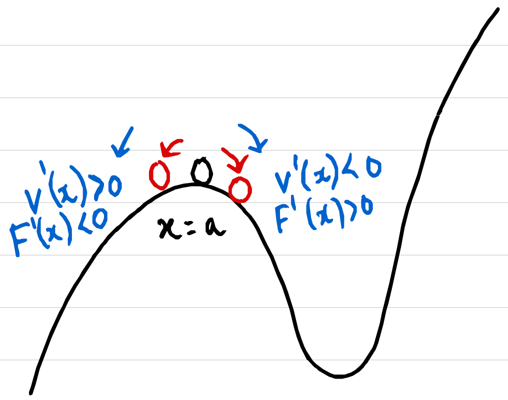
so force is directed away from equilibrium and will move towards b

- Similarly if initially pertubed particle is located at point $x<a$ then

$$V'(x)>0 \Rightarrow F(x)<0 \quad \text{for } x<a$$

Then so that again force is directed away from equilibrium and particle will escape towards ∞

Thus $x=a$ is an unstable equilibrium



Thus equilibrium is unstable if it is a local maximum of $V(x)$ i.e.

$$V''(x)>0$$

at the equilibrium

When a critical point of V $(V'(x) = 0)$ is neither a maximum or minimum, but a point of inflection,

↳ it can be shown that in this case the equilibrium is unstable.
(Again because these are initial perturbations for which force is directed away from equilibrium point.

Defn: Formal Defn of Stability:

Example problem 8: (motion in a potential):

Consider mass m of the particle moving on a straight line under the action force with potential

$$V(x) = \frac{kx}{x^2+a^2}$$

where k and a are positive constants.

(a) Find equilibria of particle.
Sketch graph of $V(x)$ and determine stability of equilibrium points.

Solution: Equilibria are critical points.

$$V'(x) = \frac{k}{x^2+a^2} - \frac{2kx}{x^2+a^2} = \frac{k(x^2+a^2)-2kx^2}{(x^2+a^2)^2}$$

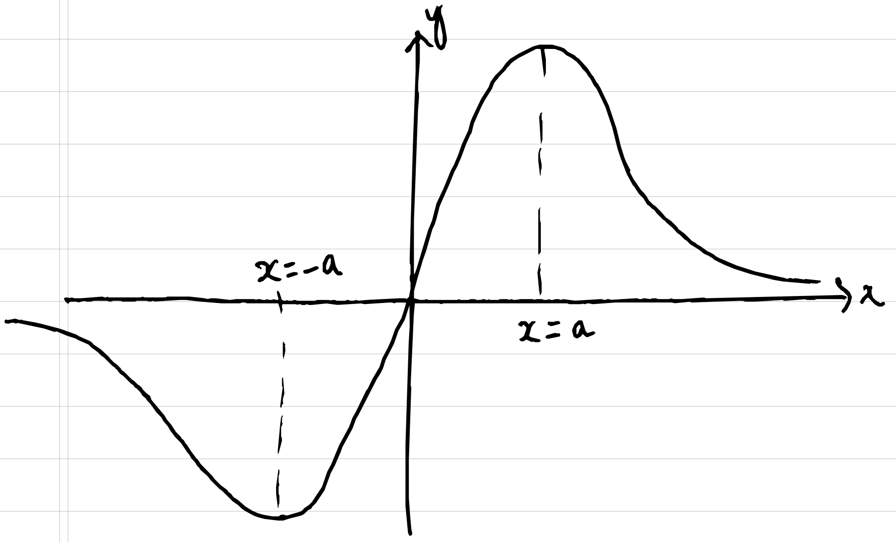
$$= \frac{k(a^2-x^2)}{x^2+a^2}$$

$$V'(x) = 0 \Rightarrow x = \pm a$$

Also we have

$$V(\pm a) = \pm \frac{k}{2a}, \quad V(0) = 0, \quad V(x) \to 0 \text{ as } x \to \pm\infty$$

Sketching graph using the information



(b) particle passes origin $x=0$ moving in positive x direction with velocity $v_0 > 0$

(i) prove that the particle will subsequently pass the point $x=a$ if and only if

$$v_0^2 > \frac{k}{ma}$$

(ii) Find a condition on v_0 such that the particle passes the point $x=-a$

Solution:

(i) particle will pass point $x=a$ if total energy E is greater than $V(a)$. Therefore

$$E > V(a) \Rightarrow \frac{mv_0^2}{2} + V(0) > V(a)$$

$$\Rightarrow \frac{mv_0^2}{2} + 0 > \frac{k}{2a}$$

$$\Rightarrow v_0^2 > \frac{k}{ma}$$

(ii) The particle will subsequently pass the point $x=-a$ if its direction of motion reverses, i.e. if total energy E is less than $V(a)$. Hence

$$\frac{mv_0^2}{2} + V(0) = \frac{mv_0^2}{2} < V(a)$$

$$\Rightarrow \frac{mv_0^2}{\cancel{2}} < \frac{k}{\cancel{2}a}$$

$$\Rightarrow v_0^2 < \frac{k}{ma}$$

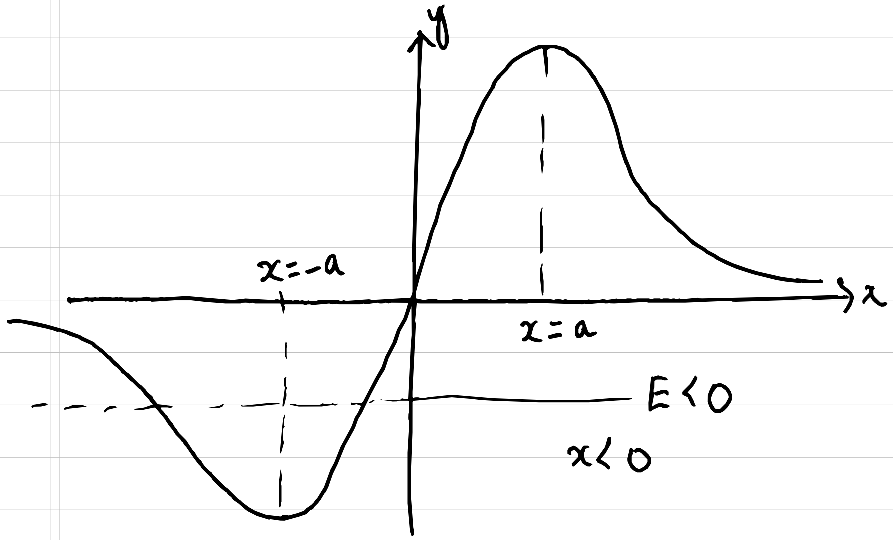
c) find initial positions and initial velocities for which particles motion is periodic

Solution: Particles motion will be periodic if $E<0$, i.e. if

$$E<0 \Rightarrow \frac{mv_0^2}{2}+V(x_0)<0$$

$$\Rightarrow \frac{mv_0^2}{2}+\frac{kx_0}{x_0^2+a^2}<0$$

From graph this inequality is only satisfied if $x_0<0$



Therefore all possible initial velocities are determined from the inequality

$$\frac{mv_0^2}{2}<\frac{-kx_0}{x_0^2+a^2} \quad \text{or}$$

$$v_0^2<-\frac{2kx_0}{m(x_0^2+a^2)}$$

Thus the set of initial co-ordinates and velocities corresponding to periodic motion is

$$x_0\in(-\infty,0) \quad (\text{since } x_0<0)$$

$$v_0\in\left(-\sqrt{\frac{-2kx_0}{m(x_0^2+a^2)}},\ \sqrt{\frac{-2kx_0}{m(x_0^2+a^2)}}\right)$$

Example problem 9: (Escape velocity):

Lets find minimum velocity needed for a body of mass m to escape earths gravity.

Let M: mass of earth

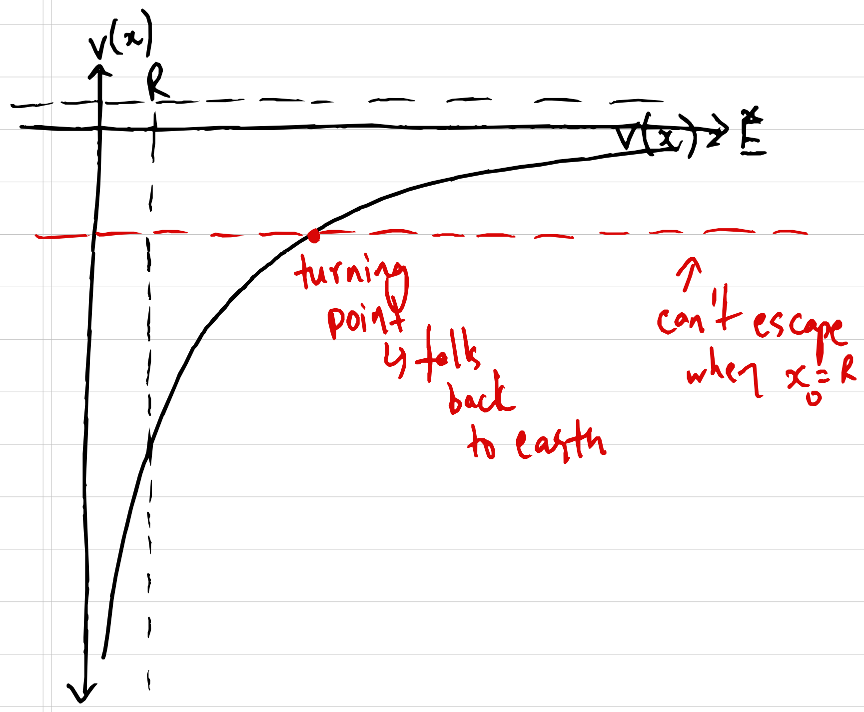
R: radius of earth

Introduce x axis that passes through centre of earth and body, origin at centre of Earth. x axis is directed from Earth to body.
We know that the Newtonian gravitational force of attraction acting on the body and the corresponding gravitational potential are

$$F(x) = -\frac{GmM}{x^2} \qquad V(x) = -\frac{GmM}{x}$$

where G is the gravitational constant.

Sketch of $V(x)$ below:



$E<0$: From the graph, it is clear that if the total energy of particle is negative and body is initially at $x_0 = R$, (i.e. at surface of earth) with velocity $v_0 > 0$, particle will reach turning point $V(x) = E$ and will fall back to earth

In order to escape earths gravity, total energy must be non-negative, $E \geq 0$.

Minimum velocity corresponds to $E = 0$

Therefore

$$E = \frac{m\dot{x}^2}{2} - \frac{GmM}{x} \Rightarrow E = \frac{mv_0^2}{2} - \frac{GmM}{x_0}$$

$$\Rightarrow E = \frac{mv_0^2}{2} - \frac{GmM}{R}$$

$$\Rightarrow 0 = \frac{mv_0^2}{2} - \frac{GmM}{R}$$

$$\Rightarrow v_0 = \sqrt{\frac{2GM}{R}}$$

Observe that gravitational force at Earths surface $(x=R)$ is $F = -mg$. So

$$-\frac{GmM}{R^2} = -mg \Rightarrow \frac{GM}{R^2} = g$$

$$\Rightarrow \frac{GM}{R \times R} = g$$

$$\Rightarrow \frac{GM}{R} = gR$$

So

$$v_0 = \sqrt{\frac{2GM}{R}} \Rightarrow v_0 = \sqrt{2gR}$$

3.8) Oscillations natural and Forced

3.8.1 Motion near a stable equilibrium

We already know that if <u>a particle in stable equilibrium</u> is disturbed slightly (say by moving it to a nearby location), <u>then it will oscillate about the position of equilibrium</u>

Consider a particle moving under the force with potential $V(x)$.

Let $x=a$ be a stable equilibrium, i.e $x=a$ is a local minimum $(V'(a)=0, V''(a)>0)$

For small $|x-a|$, we expand the potential energy $V(x)$ into Taylor series about the point $x=a$. and retain only first 3 terms:

$$V(x) \approx V(a) + (x-a)V'(a) + \frac{(x-a)^2}{2} V''(a)$$

Since $x=a$ is a point of local minimum of $V(x)$ we have $V'(a)=0$, $V''(a)>0$. Hence

$$V(x) \approx V(a) + \frac{(x-a)^2}{2} V''(a)$$

$$\Rightarrow V'(x) \approx V''(a)(x-a)$$

Since $V'(x) = -F(x)$

Equation of motion becomes

$$m\ddot{x} = -V''(x)(x-a)$$

or

$$\ddot{x} = -w^2(x-a)$$

with $w^2 \equiv \frac{V''(a)}{m}$

Convenient to introduce new variable $\underline{z = x-a}$.

↳ this change in variable represents the shift of origin to the point $x=a$.

Defn: Simple Harmonic Oscillators

The equation of motion is

$$\boxed{\ddot{z} + w^2 z = 0}$$

which is the eqn of motion of a simple harmonic oscillator.

It undergoes <u>simple harmonic motion</u> with <u>general solution</u>

$$z(t) = c_1 \sin wt + c_2 \cos wt$$

where c_1, c_2 are arbitrary constants. These constants can be determined from suitable initial conditions.

The general solution can be written in several <u>equivalent forms</u>

For example it can be written as

$$z(t) = A\sin(wt + \delta)$$

where A and δ are constants, $A \geq 0$, $0 \leq \delta \leq 2\pi$
Using the trigonometric identity

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

giving us

$$z(t) = A\sin wt\cos\delta + A\cos wt\sin\delta$$

Comparing this with the first one, we get

$$c_1 = A\cos\delta \qquad c_2 = A\sin\delta$$

A, and δ can be found in terms of c_1 and c_2.
Summing the squares of the equation yields

$$A^2 = c_1^2 + c_2^2 \Rightarrow A = \sqrt{c_1^2 + c_2^2}$$

Then if $A \neq 0$ then

$$\cos\alpha = \frac{c_1}{\sqrt{c_1^2+c_2^2}} \qquad \sin\delta = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$$

give a unique value for $\delta \in [0, \delta)$

It follows from the eqn

$$z(t) = A\sin(wt + \delta)$$

that the motion is periodic, i.e. there is a constant T such that $\underline{\underline{z(t+T) = z(t)}}$.

Indeed we have

$$z(t+T) = A\sin(w(t+T) + \delta)$$

$$= A\sin(wt + \delta + wT)$$

Since sine is 2π periodic

$$z(t+T) = A\sin(w(t+T) + \delta)$$

$$= A\sin(wt + \delta + wT) = z(t) = A\sin(wt+\delta)$$

$$\Rightarrow \quad wT = 2\pi \Rightarrow T = \frac{2\pi}{w}$$

Defn: Period, frequency, angular frequency and phase:

$T = 2\pi/w$ is the period of oscillation

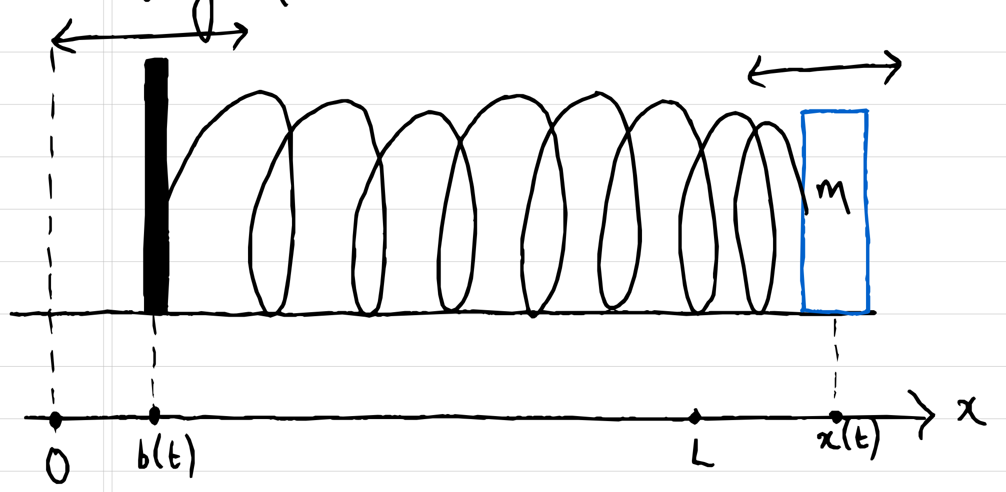
The inverse of period: $w/2\pi$ is the number of complete oscillations performed per unit time

↳ This is called frequency and w is called the angular frequency.

Constant δ in eqn $A\sin(wt+\delta)$ is called the phase constant.

3.8.2 Forced Oscillations

Consider the motion depicted in the following diagram:



A body of mass m lies on a flat smooth surface (no friction between body and surface) and attached to the right end of a weightless spring of natural length L

The left end of spring is attached to a wall which performs harmonic oscillations such that it is parallel to the surface with origin at mean position of the wall

Let $b(t)$ be the position of the wall
Let $x(t)$ be the position of the body

Then length of spring at time t is

$$x(t)-b(t).$$

We know that the elastic force acting on the body is proportional to the difference between the current length and its natural length.
Therefore the force is given by

$$F(x) = -k\left(x(t)-b(t)-L\right)$$

So equation of motion is

$$m\ddot{x} = -k\left(x(t)-b(t)-L\right)$$

where k is the elastic constant of the spring
Infact this equation is valid for arbitrary motion of the wall but we will restrict this to the case of harmonic oscillations

So we assume that

$$b(t) = \alpha \sin \Omega t \quad \alpha>0, \Omega>0$$

The equation becomes

$$m\ddot{x} = -k\left(x(t)-\alpha\sin\Omega t - L\right)$$

$$\Rightarrow \quad m\ddot{x} = -k\left[x(t)-L\right] + k\alpha \sin \Omega t$$

Note that the effect of the vibration of the wall is described by an additional time dependant force in the equation

It is convenient to introduce another new variable

$$z(t) = x(t) - L$$

and to divide the equation of motion by m

$$\ddot{z} + w^2 z = w^2 \alpha \sin \Omega t$$

where $w^2 = k/m$ is the natural angular frequency of small oscillations of mass m attached to one end of the spring whose other end is fixed

Solving (non homogeneous 2nd order ODE)

$$\ddot{z} + w^2 z = w^2 \alpha \sin \Omega t$$

1) Find $z_h(t)$: soln to homogeneous eqn:

$$\ddot{z} + w^2 z = 0$$

Ansatz: Assume soln of form: $z = e^{\lambda t}$

Auxillary eqn:

$$\lambda^2 + w^2 = 0 \Rightarrow \lambda = \pm w i$$

So $z_h(t) = c_1 \sin wt + c_2 \cos wt$

which can be written in form

$$z_h(t) = A \sin(wt + \delta)$$

2) Find $z_p(t)$ particular solution:

Assume soln of form

$$z(t) = A \sin \Omega t + B \cos \Omega t$$

$$\Rightarrow \dot{z}(t) = A\Omega \cos \Omega t - B\Omega \sin \Omega t$$

$$\Rightarrow \ddot{z}(t) = -A\Omega^2 \sin \Omega t - B\Omega^2 \cos \Omega t$$

Substituting

$$-A\Omega^2 \sin \Omega t - B\Omega^2 \cos \Omega t + w^2 A \sin \Omega t + w^2 B \cos \Omega t = w^2 \alpha \sin \Omega t$$

$$\Rightarrow \sin \Omega t(-A\Omega^2 + w^2 A) + \cos \Omega t(-B\Omega^2 + w^2 B) = w^2 \alpha \sin \Omega t$$

comparing coefficients

$$-A\Omega^2 + w^2 A = w^2 \alpha$$

$$\Rightarrow A(w^2 - \Omega^2) = w^2 \alpha \Rightarrow A = \frac{w^2 \alpha}{w^2 - \Omega^2}$$

$$-B\Omega^2 + w^2 B = 0 \Rightarrow B = 0 \quad (\text{assume } w^2 \neq \Omega^2)$$

So, $$z_p(t) = \frac{w^2 \alpha}{w^2 - \Omega^2} \sin \Omega t$$

So general solution is

$$z(t) = \frac{w^2 \alpha}{w^2 - \Omega^2} \sin \Omega t + A \sin(wt + \delta)$$

Defn: Natural and Forced Oscillations

The second term in the solution

$$z(t) = \frac{w^2\alpha}{w^2-\Omega^2}\sin\Omega t + A\sin(wt+\delta)$$

represents the solution to the homogeneous eqn which corresponds to free (natural) oscillations

The first term is produced by the periodic force on the right side of eqn

$$\ddot{z} + w^2 z = w^2\alpha\sin\Omega t$$

and describes the forced oscillations.

Choosing initial conditions

$$z(0) = 0 \ , \ \dot{z}(0) = \Omega\frac{w^2\alpha}{w^2-\Omega^2}$$

we can get rid of the second term.

$$z(0) = A\sin(\delta)$$

$$\Rightarrow A\sin(\delta) = 0 \qquad (*1)$$

$$\dot{z}(0) = \cancel{\Omega\frac{w^2\alpha}{w^2-\Omega^2}} + A\cos(\delta) = \cancel{\frac{\Omega w^2\alpha}{w^2-\Omega^2}}$$

$$\Rightarrow \quad A\cos(\delta) = 0 \qquad (*2)$$

By (*1) and (*2) we conclude $A=0$.

So

$$z(t) = \frac{w^2\alpha}{w^2-\Omega^2}\sin\Omega t$$

which represents forced oscillations.

Evidently the solution is only valid for

$$w^2 \neq \Omega^2$$

The solution

$$z(t) = \frac{w^2\alpha}{w^2-\Omega^2}\sin\Omega t \qquad (*1)$$

can be written in the form

$$z(t) = \bar{A}\sin(\Omega t + \bar{\delta}) \qquad (*2)$$

where $\bar{A} \geq 0$ is the amplitude of the forced oscillations and $\bar{\delta}$ is its phase $(\bar{\delta} \in [0, 2\pi))$

- If $\Omega^2 < w^2$ then comparing $(*1)$ and $(*2)$ we get the amplitude and phase constant of forced oscillations:

$$\bar{A} = \frac{w^2\alpha}{w^2-\Omega^2} \quad \text{and} \quad \bar{\delta} = 0$$

So the phase of the forcing term in

$$\ddot{z} + w^2 z = \underline{w^2\alpha\sin\Omega t}$$

coincide with the motion of forced oscillations given by $z(t)$

↳ which means the body follows the wall. (when the wall moves to the left (or right) the wall does the same).

- If $\Omega^2 < w^2$ then by comparing $z(t) = \frac{w^2\alpha}{w^2-\Omega^2}\sin\Omega t$

and $z(t) = \bar{A}\sin(\Omega t + \bar{\delta})$

we have

$$z(t) = \frac{w^2\alpha}{w^2-\Omega^2}\sin\Omega t = \frac{-w^2\alpha}{\Omega^2-w^2}\sin(\Omega t)$$

$$= \frac{w^2\alpha}{\Omega^2-w^2}\sin(\Omega t + \pi)$$

$$\Rightarrow z(t) = \frac{w^2\alpha}{\Omega^2-w^2}\sin(\Omega t + \pi)$$

It follows that

$$\bar{A} = \frac{w^2\alpha}{\Omega^2-w^2} \quad \text{and} \quad \bar{\delta} = \pi$$

i.e. phase of the forcing term and phase of the motion of forced oscillation given by $z(t)$ differs by π

↳ This means that when the wall is moving to the left, the body is moving to the right and vice versa.

Defn: Resonance

In both case, the amplitude of forced oscillations increases without limit as $\Omega^2 \to w^2$

Ω: angular frequency of forced oscillations

w: angular frequency of natural oscillations

As frequency (angular) of forced approaches the angular frequency of masses natural frequency (angular) the bodies amplitude increases without limit.

This phenomenon is called resonance

So the solution given is only valid for $w^2 \neq \Omega^2$. The case $w^2 = \Omega$ requires seperate analysis.

We know need a different particular solution (since $\Omega^2 = w^2 \Rightarrow \Omega = w$ so our original particular solution was part of homogeneous one)

Assume solution of form

$$z_p = C_1 t \sin \Omega t + C_2 \cos \Omega t.$$

Substituting into $\ddot{z} + w^2 z = w^2 \alpha \sin \Omega t$ yields

$$2\Omega(C_1 \cos \Omega t + C_2 \sin \Omega t) = \Omega^2 \alpha \sin \Omega t$$

$\Rightarrow C_1 = 0$ and $2\Omega C_2 = -\Omega^2 \alpha$

Hence $C_2 = -\frac{\Omega \alpha}{2}$.

We obtain

$$z_p = -\frac{\Omega \alpha}{2} t \cos \Omega t.$$

By initial conditions, homogeneous soln disappears

So general solution can be written as

$$z_p = \bar{A}(t) \sin(\Omega t + \bar{\delta}) \quad \text{where}$$

$$\bar{A}(t) \equiv \frac{\Omega \alpha}{2} t, \quad \bar{\delta} \equiv \frac{3\pi}{2}$$

Thus in case of resonance amplitude forced oscillations grows linearly in time in this case.